

# CONDITIONAL QUASI-GREEDY BASES IN HILBERT AND BANACH SPACES

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**ABSTRACT.** For quasi-greedy bases  $\mathcal{B}$  in Hilbert spaces, we give an improved bound of the associated conditionality constants  $k_N(\mathcal{B}) = O(\log N)^{1-\varepsilon}$ , for some  $\varepsilon > 0$ , answering a question by Temlyakov. We show the optimality of this bound with an explicit construction, based on a refinement of the method of Olevskii. This construction leads to other examples of quasi-greedy bases with large  $k_N$  in Banach spaces, which are of independent interest.

## 1. INTRODUCTION

The concept of quasi-greedy basis evolved from the analysis of thresholding algorithms for non-linear  $N$ -term approximation in Banach spaces; see e.g. [15] for a detailed presentation and background. In recent years it has attracted attention from both, the approximation theory and the Banach space point of view.

Let us recall the relevant definitions and standard notation. For a (normalized) basis  $\{\mathbf{e}_j\}_{j=1}^\infty$  in a Banach space  $\mathbb{X}$  and  $N = 1, 2, \dots$  we consider non-linear operators  $G_N$  as follows

$$x = \sum_{j=1}^{\infty} a_j \mathbf{e}_j \in \mathbb{X} \longmapsto G_N(x) = \sum_{j \in \Lambda} a_j \mathbf{e}_j,$$

where  $\Lambda$  is *any*  $N$ -element subset of  $\{1, 2, \dots\}$  such that  $\min_{j \in \Lambda} |a_j| \geq \max_{j \notin \Lambda} |a_j|$ . Then  $\{\mathbf{e}_j\}$  is called a *quasi-greedy basis* if for any  $x \in \mathbb{X}$  and any choice of  $G_N$ 's we have  $\lim_{N \rightarrow \infty} \|x - G_N(x)\| = 0$ , that is the series defining  $x$  converges in norm after decreasing rearrangement of their summands. It is known (see [18]) that this is equivalent to

$$\|G_N x\| \leq K \|x\|, \quad \forall x \in \mathbb{X}, \quad N = 1, 2, \dots \quad (1.1)$$

for some (smallest) constant  $K$ , which we assume fixed throughout the paper. In particular, every unconditional basis is quasi-greedy, but there exist also examples of *conditional* quasi-greedy bases [12, 18, 4, 13, 9, 6]. In this paper we shall be interested in the latter.

Associated with a basis  $\mathcal{B} = \{\mathbf{e}_j\}$  in  $\mathbb{X}$ , we consider the sequence

$$k_N = k_N(\mathcal{B}) := \sup_{|A| \leq N} \|S_A\|, \quad N = 1, 2, \dots$$

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where  $S_A : \mathbb{X} \rightarrow \mathbb{X}$  denotes the projection operator  $S_A(x) = \sum_{j \in A} a_j(x) \mathbf{e}_j$ . Generally speaking, the constants  $k_N$  quantify the conditionality of the basis  $\mathcal{B}$ . In fact,  $\mathcal{B}$  is unconditional if and only if  $k_N(\mathcal{B}) = O(1)$ .

In approximation theory  $k_N$  can also be used to quantify the performance of greedy algorithms with respect to the best  $N$ -term approximation from  $\{\mathbf{e}_j\}$ ; that is, if  $C_N$  denotes the smallest constant such that

$$\|x - G_N x\| \leq C_N \inf \left\{ \|x - \sum_{j \in A} c_j \mathbf{e}_j\| : c_j \in \mathbb{C}, |A| \leq N \right\}, \quad \forall x \in \mathbb{X},$$

then it is proved in [8, 17] that  $C_N \approx k_N$  when  $\{\mathbf{e}_j\}$  is an *almost-greedy* basis of  $\mathbb{X}$  (i.e. quasi-greedy and democratic<sup>1</sup>). Thus, in this case the constants  $k_N$  also give information on the rate of convergence of greedy algorithms.

It is known that for quasi-greedy bases in Banach spaces one has

$$k_N = O(\log N)$$

(see [4, Lemma 8.2]), and this bound is actually attained in some Banach spaces [8]. It was asked in [16, p. 335] whether this bound is optimal or could be improved in the case of Hilbert spaces. Our first result answers this question.

**Theorem 1.1.** *Let  $\mathbb{H}$  be a Hilbert space and  $\{\mathbf{e}_j\}$  a quasi-greedy (normalized) basis with constant  $K$ . Then, there exists  $\alpha = \alpha(K) < 1$  and  $c > 0$  such that*

$$k_N(\{\mathbf{e}_j\}) \leq c (\log N)^\alpha, \quad \forall N \in \mathbb{N}. \quad (1.2)$$

Moreover, if  $\{\mathbf{e}_j\}$  is besselian or hilbertian then one can choose  $\alpha < \frac{1}{2}$  in (1.2).

Recall that  $\{\mathbf{e}_j\}$  is *besselian* if  $\sum_j |a_j|^2 \leq C \|\sum_j a_j \mathbf{e}_j\|_{\mathbb{X}}^2$  for all finitely supported scalars  $(a_j)$ , and is called *hilbertian* if the converse inequality  $\sum_j |a_j|^2 \geq C \|\sum_j a_j \mathbf{e}_j\|_{\mathbb{X}}^2$  holds.

Our second result proves that the bound obtained in (1.2) is actually optimal.

**Theorem 1.2.** *For every  $\alpha < 1$ , there exists a quasi-greedy basis in  $\mathbb{H}$  and a constant  $c_\alpha > 0$  such that*

$$k_N \geq c_\alpha (\log N)^\alpha, \quad N = 1, 2, \dots \quad (1.3)$$

*If  $\alpha < 1/2$ , then the basis can be chosen to be in addition besselian (or hilbertian).*

Theorem 1.1 is shown in §2, with an explicit expression for  $\alpha = \alpha(K)$  given in (2.9). In the proof we make use of the inner product structure of  $\mathbb{H}$ , although the argument can be adapted to other settings, such as  $L^p$  spaces,  $1 < p < \infty$ , for which (1.2) is also true if  $\{\mathbf{e}_j\}$  is quasi-greedy (see Appendix II).

Theorem 1.2 is shown in §3. The proof is based on a construction due to Olevskii, which was developed in [18] to produce conditional quasi-greedy bases in Banach spaces. This construction has an independent interest, and is stated as Theorem 3.1 below. Its proof contains new ideas compared to [18, Theorem 2]. Namely, we refine the method so that besselian assumptions are not needed, and moreover to obtain a basis which is almost-greedy and has largest possible  $k_N$ . Only in this way we can

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<sup>1</sup>In Hilbert spaces, quasi-greedy bases are always democratic [18], so both concepts coincide.

reach the optimal bounds in (1.3)<sup>2</sup>. We also apply this construction to obtain new examples of almost-greedy bases in Banach spaces with  $k_N \approx \log N$ .

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## 2. PROOF OF THEOREM 1.1

Below we identify  $x = \sum_{j=1}^{\infty} a_j(x) \mathbf{e}_j \in \mathbb{H}$  with the coefficient sequence  $(a_j)_{j=1}^{\infty}$ , so we write  $\text{supp } x = \{j \in \mathbb{N} : a_j(x) \neq 0\}$ . We shall use the following definition.

**Definition 2.1.** Let  $x, y \in \mathbb{H}$ . We say that  $x \succ y$  if

- (i)  $\text{supp } x \cap \text{supp } y = \emptyset$
- (ii)  $\min_{i \in \text{supp } x} |a_i(x)| \geq \max_{j \in \text{supp } y} |a_j(y)|$

The key result is the following lemma. Recall that the quasi-greedy constant  $K$  was defined in (1.1).

**Lemma 2.2.** There exists  $\delta = \delta(K) \in [0, 1)$  such that, for all  $x \succ y$

$$(1 - \delta)(\|x\|^2 + \|y\|^2) \leq \|x + y\|^2 \leq (1 + \delta)(\|x\|^2 + \|y\|^2). \quad (2.1)$$

In fact, (2.1) holds with  $\delta = \sqrt{1 - \frac{1}{K^2}}$ .

**PROOF:** Let  $\gamma \in \mathbb{R}$  with  $|\gamma| \leq 1$ . Then (1.1) implies

$$\|x\|^2 \leq K^2 \|x - \gamma y\|^2 = K^2 (\|x\|^2 + \gamma^2 \|y\|^2 - 2\gamma \Re \langle x, y \rangle).$$

Hence we have the inequality

$$\|y\|^2 \gamma^2 - 2\gamma \Re \langle x, y \rangle + (1 - \frac{1}{K^2}) \|x\|^2 \geq 0, \quad \forall \gamma \in [-1, 1].$$

This inequality holds with  $y$  replaced by  $e^{i\theta} y$ , for any  $\theta \in \mathbb{R}$ , so we also have

$$\|y\|^2 \gamma^2 - 2\gamma |\langle x, y \rangle| + (1 - \frac{1}{K^2}) \|x\|^2 \geq 0, \quad \forall \gamma \in [-1, 1]. \quad (2.2)$$

Substituting  $\gamma = \sqrt{1 - \frac{1}{K^2}}$  into (2.2) we obtain

$$2|\langle x, y \rangle| \leq \sqrt{1 - \frac{1}{K^2}} (\|x\|^2 + \|y\|^2), \quad (2.3)$$

from which (2.1) follows easily with  $\delta = \sqrt{1 - \frac{1}{K^2}}$ .  $\square$

As a special case of the lemma we obtain

**Corollary 2.3.** If  $\{\mathbf{e}_j\}$  is a quasi-greedy (normalized) basis in  $\mathbb{H}$  such that

$$\|G_N x\| \leq \|x\|, \quad \forall x \in \mathbb{H}, N = 1, 2, \dots$$

then,  $\{\mathbf{e}_j\}$  is an orthonormal basis.

**PROOF:** Applying Lemma 2.2 to  $x = \mathbf{e}_i$ ,  $y = \mathbf{e}_j$  with  $i \neq j$  and  $K = 1$ , we obtain  $\langle \mathbf{e}_i, \mathbf{e}_j \rangle = 0$ .  $\square$

<sup>2</sup>The construction in [18] would only lead to  $k_N \gtrsim (\log \log N)^{\alpha}$ .

An iteration of the previous lemma leads to the following. We denote, for  $\gamma \in \mathbb{R}$ ,  $\lceil \gamma \rceil = \min\{k \in \mathbb{Z} : \gamma \leq k\}$ .

**Lemma 2.4.** *Let  $\delta = \delta(K)$  be as in Lemma 2.2, then for all  $x_1 \succcurlyeq x_2 \succcurlyeq \dots \succcurlyeq x_m$  with pairwise disjoint supports we have*

$$(1 - \delta)^{\lceil \log_2 m \rceil} \sum_{j=1}^m \|x_j\|^2 \leq \|x_1 + \dots + x_m\|^2 \leq (1 + \delta)^{\lceil \log_2 m \rceil} \sum_{j=1}^m \|x_j\|^2. \quad (2.4)$$

**PROOF:** We shall prove the result for  $2^{n-1} < m \leq 2^n$  by induction in  $n = \lceil \log_2 m \rceil$ . The case  $n = 1$  corresponds to (2.1). Assume (2.4) holds for  $m \leq 2^n$ , and we shall verify it for  $2^n < m \leq 2^{n+1}$ . Call  $x' = \sum_{1 \leq j \leq 2^n} x_j$  and  $y' = \sum_{2^n < j \leq m} x_j$ . Since  $x' \succcurlyeq y'$ , Lemma 2.2 gives

$$\begin{aligned} \left\| \sum_{j=1}^m x_j \right\|^2 &= \|x' + y'\|^2 \leq (1 + \delta) \left( \|x'\|^2 + \|y'\|^2 \right) \\ &= (1 + \delta) \left( \|x_1 + \dots + x_{2^n}\|^2 + \|x_{2^n+1} + \dots + x_m\|^2 \right) \\ &\leq (1 + \delta)^{n+1} \sum_{j=1}^m \|x_j\|^2, \end{aligned}$$

using the induction hypothesis in the last step. The inequality from below is similar.  $\square$

From these two lemmas, the proof of Theorem 1.1 is similar to [8, Theorem 5.1] (see also [4, 6]).

### PROOF of Theorem 1.1:

Let  $A \subset \mathbb{N}$  with  $|A| = N \geq 2$ . We must show that, for all  $x = \sum_i a_i \mathbf{e}_i \in \mathbb{H}$  then

$$\|S_A(x)\| \leq c (\log N)^\alpha \|x\|, \quad (2.5)$$

for some  $\alpha < 1$  (independent of  $x$  and  $N$ ). By scaling we may assume  $\max_i |a_i| = 1$  (which by (1.1) implies  $\|x\| \geq 1/K$ ).

Let  $m = \lceil \log_2 N \rceil$ , so that  $2^{m-1} < N \leq 2^m$ . For  $\ell = 1, \dots, m$ , we define

$$F_\ell = \{j : 2^{-\ell} < |a_j| \leq 2^{-(\ell-1)}\} \quad \text{and} \quad F_{m+1} = \{j : |a_j| \leq 2^{-m}\}.$$

Next write  $A$  as a disjoint union of the sets  $A_\ell = A \cap F_\ell$ ,  $\ell = 1, \dots, m+1$ . Clearly

$$\|S_{A_{m+1}}x\| \leq \sum_{i \in A_{m+1}} |a_i| \|\mathbf{e}_i\| \leq 2^{-m} N \leq 1 \leq K \|x\|. \quad (2.6)$$

For the other terms we shall use Lemmas 5.2 and 5.3 in [8], which give

$$\|S_{A_\ell}x\| \leq c_1 \|S_{F_\ell}(x)\| \leq c_2 \|x\|,$$

with  $c_1 = 64K^3$  and  $c_2 = 128K^4$ . Now, Lemma 2.4 gives

$$\left\| \sum_{\ell=1}^m S_{A_\ell}x \right\|^2 \leq (1 + \delta)^{\lceil \log_2 m \rceil} \sum_{\ell=1}^m \|S_{A_\ell}x\|^2 \leq c_1^2 (1 + \delta)^{\lceil \log_2 m \rceil} \sum_{\ell=1}^m \|S_{F_\ell}x\|^2. \quad (2.7)$$

We now have two possible approaches. In the first approach we use the lower bound in (2.4), so that (2.7) becomes

$$\left\| \sum_{\ell=1}^m S_{A_\ell} x \right\|^2 \leq c_1^2 \left( \frac{1+\delta}{1-\delta} \right)^{\lceil \log_2 m \rceil} \left\| \sum_{\ell=1}^m S_{F_\ell} x \right\|^2 \leq c_3^2 \left( \frac{1+\delta}{1-\delta} \right)^{\lceil \log_2 m \rceil} \|x\|^2,$$

with  $c_3 = Kc_1 = 64K^4$ . Observe that

$$\left( \frac{1+\delta}{1-\delta} \right)^{\log_2 m} = 2^{\log_2 m \log_2 \frac{1+\delta}{1-\delta}} = m^{\log_2 \frac{1+\delta}{1-\delta}} = m^{2\alpha_1}$$

if we take  $\alpha_1 = \frac{1}{2} \log_2 \frac{1+\delta}{1-\delta}$ . Notice however that  $\alpha_1 < 1$  if and only if  $\delta < 3/5$ , so this approach is not good for  $\delta$  close to 1 (ie, when  $K$  is very large).

A second approach for (2.7) consists in estimating each  $\|S_{A_\ell} x\| \leq c_2 \|x\|$ . Then

$$\left\| \sum_{\ell=1}^m S_{A_\ell} x \right\|^2 \leq c_2^2 (1+\delta)^{\lceil \log_2 m \rceil} m \|x\|^2. \quad (2.8)$$

Now we can write

$$(1+\delta)^{\log_2 m} m = 2^{\log_2 m \log_2 (1+\delta)} m = m^{1+\log_2 (1+\delta)} = m^{2\alpha_2},$$

if we choose  $\alpha_2 = (1+\log_2(1+\delta))/2$ . Notice that this time  $\alpha_2 < 1$ , but it may happen that  $\alpha_2 > \alpha_1$  if  $\delta < 1/2$ . In that case (ie, when  $K$  is close to 1) the former choice is slightly better.

Combining the two approaches, and using also (2.6), we see that (1.2) holds with

$$\alpha = \frac{1}{2} \min \left\{ \log_2 \frac{1+\delta}{1-\delta}, 1 + \log_2(1+\delta) \right\}, \quad (2.9)$$

with the minimum attained in the first number for  $\delta \leq \frac{1}{2}$ , and in the second number for  $\delta \geq \frac{1}{2}$ . Recall also from Lemma 2.2 that  $\delta = \sqrt{1 - 1/K^2}$ .

Finally, suppose that the basis  $\{\mathbf{e}_j\}$  is not only quasi-greedy, but also *besselian*. Quasi-greediness implies that  $\ell^{2,1} \hookrightarrow \mathbb{H}$  ([18, Thm 3]), so we can estimate for each  $\ell$

$$\|S_{F_\ell} x\| \lesssim 2^{-\ell} |F_\ell|^{1/2} \leq \left( \sum_{j \in F_\ell} |a_j|^2 \right)^{1/2}.$$

Inserting this into (2.7) and using that  $\mathbb{H} \hookrightarrow \ell^2$  (from the besselian assumption), we obtain

$$\left\| \sum_{\ell=1}^m S_{A_\ell} x \right\|^2 \lesssim (1+\delta)^{\lceil \log_2 m \rceil} \sum_j |a_j|^2 \lesssim m^{\log_2(1+\delta)} \|x\|^2.$$

Thus, (1.2) holds with  $\alpha = \frac{1}{2} \log_2(1+\delta)$ , which is always a real number  $< 1/2$ . The same bound holds when  $\{\mathbf{e}_j\}$  is *hilbertian*, since in this case the dual basis  $\{\mathbf{e}_j^*\}$  is besselian in  $\mathbb{H}^*$  (and also quasi-greedy, by [5]), while  $k_N$  is the same for both bases.  $\square$

## 3. THE PROOF OF THEOREM 1.2

**3.1. A general construction of quasi-greedy bases.** The next result gives a general method to produce quasi-greedy bases in Banach spaces. As mentioned in the introduction, it is also an improvement over the statement in [18, Theorem 2].

**Theorem 3.1.** *Let  $\mathcal{X}$  be a Banach space with a (normalized) basis  $\mathfrak{X} = \{x_k\}_{k=1}^\infty$ . Then, the space<sup>3</sup>  $\mathcal{X} \oplus \ell^2$  has a quasi-greedy basis  $\Psi$ .*

Moreover,

- (i)  $\Psi$  is democratic and  $\|\sum_{\lambda \in \Lambda} \psi_\lambda\| \approx |\Lambda|^{1/2}$ .
- (ii) if the basis  $\mathfrak{X}$  is besselian (or hilbertian), so is  $\Psi$ .
- (iii) if the basis  $\mathfrak{X}$  has the property that, for some  $c > 0$  and every  $N = 1, 2, \dots$

$$\exists x \in \mathcal{X} \text{ and } A \subset \{1, \dots, N\} \text{ such that } \|S_A x\| \geq c k_N(\mathfrak{X}) \|x\|, \quad (3.1)$$

then the quasi-greedy basis  $\Psi$  satisfies

$$k_N(\Psi) \gtrsim k_{\log N}(\mathfrak{X}), \quad N = 2, 3, \dots$$

**PROOF:** Write  $\mathfrak{X} = \{x_k\}_{k=1}^\infty$  for the basis in  $\mathcal{X}$ , and  $\{e_j\}_{j=1}^\infty$  for the canonical orthonormal basis in  $\ell^2$ . In the direct sum  $\mathcal{X} \oplus \ell^2$ , consider the system of vectors  $\Upsilon$  given by

$$x_1, e_1; x_2, e_2, \dots, e_{n_3}; x_3, e_{n_3+1}, \dots, e_{n_4}; \dots \quad (3.2)$$

for a suitable increasing sequence  $n_k$ . Here we choose  $n_1 = 0$ ,  $n_2 = 1$  and  $n_{k+1} = n_k + 2^k - 1$ , so that each block  $\Upsilon_k = \{x_k, e_{n_k+1}, \dots, e_{n_{k+1}}\}$  generates a subspace  $\mathbb{H}_k$  of dimension  $N_k := n_{k+1} - n_k + 1 = 2^k$ , of which  $\Upsilon_k$  is a natural orthonormal basis. We rename this basis as

$$\Upsilon_k = \{g_{k,1}, \dots, g_{k,2^k}\},$$

and write the system in (3.2) as  $\Upsilon = \bigcup_{k=1}^\infty \Upsilon_k$ . The next lemma follows from elementary Banach space theory.

**Lemma 3.2.** *The system  $\Upsilon$  in (3.2) is a basis in  $\mathcal{X} \oplus \ell^2$ . Moreover, if  $\mathfrak{X}$  is besselian (or hilbertian), so is  $\Upsilon$ .*

We now use the Olevskii construction; see [18]. For each  $k$ , let  $A = A^{(k)}$  denote the matrix in  $SO(2^k, \mathbb{R})$  with entries given by the Haar basis in  $\mathbb{R}^{2^k}$ , ie

$$A^{(k)} = \text{col} \{h_0, h_1, \dots, h_{2^k-1}\}$$

where  $h_0 = 2^{-k/2}$ . In each subspace  $\mathbb{H}_k$  we define a new orthonormal basis  $\Psi_k = \{\psi_{k,1}, \dots, \psi_{k,2^k}\}$ , by letting

$$\begin{pmatrix} \psi_{k,1} \\ \vdots \\ \psi_{k,2^k} \end{pmatrix} = \begin{pmatrix} A^{(k)} \end{pmatrix} \begin{pmatrix} g_{k,1} \\ \vdots \\ g_{k,2^k} \end{pmatrix} \quad (3.3)$$

That is,

$$\psi_{k,\ell} = 2^{-k/2} x_k + \sum_{m=2}^{2^k} a_{\ell,m} g_{k,m}, \quad \ell = 1, \dots, 2^k. \quad (3.4)$$

From the orthonormality of  $\Psi_k$  and Lemma 3.2 it easily follows that

<sup>3</sup>Endowed with the norm  $\|x + y\|_{\mathcal{X} \oplus \ell^2}^2 = \|x\|_{\mathcal{X}}^2 + \|y\|_{\ell^2}^2$ .

**Lemma 3.3.** *The system  $\Psi = \bigcup_{k=1}^{\infty} \Psi_k$  is a basis of  $\mathcal{X} \oplus \ell^2$ . Again, if  $\mathfrak{X}$  is besselian (or hilbertian), so is  $\Psi$ .*

The key step of Theorem 3.1 is to establish the quasi-greediness of  $\Psi$ . For this we need to refine the analysis of Olevskii construction given in [18]. Notice that we do not require the basis  $\mathfrak{X}$  to be besselian in  $\mathcal{X}$ .

**Lemma 3.4.**  *$\Psi$  is a quasi-greedy basis of  $\mathcal{X} \oplus \ell^2$ , that is*

$$\|G_N(z)\| \leq C \|z\|, \quad \forall z \in \mathcal{X} \oplus \ell^2, \quad N = 1, 2, \dots \quad (3.5)$$

**PROOF:** Let  $z = \sum_{k=1}^{\infty} \sum_{\ell=1}^{2^k} c_{k,\ell} \psi_{k,\ell}$ , and  $\Lambda := \text{supp } G_N(z)$ . We use the notation  $P_{\mathcal{X}}, P_{\mathcal{H}}$  for the natural projections onto  $\mathcal{X}$  and  $\mathcal{H} = \ell^2$  respectively, and  $S_{\Lambda}$  for the projection onto  $\text{span}\{\psi_{\lambda}\}_{\lambda \in \Lambda}$ . We also write  $\Lambda_k = \{\ell : (k, \ell) \in \Lambda\}$ .

We need to show that

$$\|S_{\Lambda}z\|^2 = \|P_{\mathcal{X}}S_{\Lambda}z\|^2 + \|P_{\mathcal{H}}S_{\Lambda}z\|^2 \leq C \|z\|^2.$$

We begin with the first summand on the left hand side; that is, we shall show that

$$\|P_{\mathcal{X}}S_{\Lambda}z\| \leq C \|z\|. \quad (3.6)$$

Let  $\alpha = \min_{(k,\ell) \in \Lambda} |c_{k,\ell}|$ , which we may assume  $\alpha > 0$  (otherwise  $G_N z = z$  and (3.5) is trivial). Fix  $M \geq 1$  to be chosen later, and notice from (3.4) that we can split

$$P_{\mathcal{X}}S_{\Lambda}(z) = \sum_{k \geq M} 2^{-k/2} \left( \sum_{\ell \in \Lambda_k} c_{k,\ell} \right) x_k + \sum_{k < M} 2^{-k/2} \left( \sum_{\ell \in \Lambda_k} c_{k,\ell} \right) x_k. \quad (3.7)$$

The first term has norm bounded by

$$\begin{aligned} \left\| \sum_{k \geq M} 2^{-k/2} \left( \sum_{\ell \in \Lambda_k} c_{k,\ell} \right) x_k \right\| &\leq \sum_{k \geq M} 2^{-k/2} \sum_{\ell \in \Lambda_k} |c_{k,\ell}| \\ &\leq \sum_{k \geq M} 2^{-k/2} \frac{1}{\alpha} \sum_{\ell \in \Lambda_k} |c_{k,\ell}|^2 \\ &\leq c 2^{-M/2} \frac{1}{\alpha} \sup_k \|P_{\mathbb{H}_k} z\|^2 \leq c' 2^{-M/2} \frac{\|z\|^2}{\alpha}. \end{aligned} \quad (3.8)$$

If  $\|z\| \leq 2\alpha$ , we can choose  $M = 1$  and we are done. Otherwise

$$\left\| \sum_{k < M} 2^{-k/2} \left( \sum_{\ell \in \Lambda_k} c_{k,\ell} \right) x_k \right\| \leq \left\| \sum_{k < M} 2^{-k/2} \left( \sum_{\ell=1}^{2^k} c_{k,\ell} \right) x_k \right\| + \left\| \sum_{k < M} 2^{-k/2} \left( \sum_{\ell \notin \Lambda_k} c_{k,\ell} \right) x_k \right\| = I + II.$$

Clearly,

$$I = \|P_{\mathcal{X}}(S_{\{k < M\}} z)\| \leq \|S_{\{k < M\}}(z)\| \leq C \|z\|$$

since  $\Psi$  is a basis. Finally, since  $\max_{(k,\ell) \notin \Lambda} |c_{k,\ell}| \leq \alpha$ ,

$$II \leq \sum_{k < M} 2^{-k/2} \sum_{\ell \notin \Lambda_k} |c_{k,\ell}| \leq \alpha \sum_{k < M} 2^{-k/2} \leq c \alpha 2^{-M/2}. \quad (3.9)$$

So we can optimize in (3.8) and (3.9) by choosing  $M$  such that  $2^M = \|z\|^2/\alpha^2$ . This proves (3.6).

Next we show that

$$\|P_{\mathcal{H}}S_{\Lambda}z\| \leq C \|z\|. \quad (3.10)$$

This would be easy to establish if we assume that  $\mathfrak{X}$  is besselian. Indeed, in that case, using the orthogonality of the spaces  $P_{\mathcal{H}}(\mathbb{H}_k)$  we can write

$$\begin{aligned} \|P_{\mathcal{H}}S_{\Lambda}z\|^2 &= \sum_k \left\| \sum_{\ell \in \Lambda_k} c_{k,\ell} P_{\mathcal{H}}(\psi_{k,\ell}) \right\|^2 \\ &\leq \sum_k \left\| \sum_{\ell \in \Lambda_k} c_{k,\ell} \psi_{k,\ell} \right\|^2 = \sum_k \sum_{\ell \in \Lambda_k} |c_{k,\ell}|^2 \\ &\leq \sum_k \sum_{\ell=1}^{2^k} |c_{k,\ell}|^2 \leq C \|z\|^2, \end{aligned} \quad (3.11)$$

where in the last inequality we would use that the basis  $\Psi$  is also besselian.

We now give a different argument which holds for general  $\mathfrak{X}$ . As before, we define  $\alpha = \min_{(k,\ell) \in \Lambda} |c_{k,\ell}|$  which we may assume  $\alpha > 0$ . We write  $z_k = P_{\mathbb{H}_k}(z) = \sum_{\ell=1}^{2^k} c_{k,\ell} \psi_{k,\ell}$  as

$$z_k = P_{\mathcal{X}}(z_k) + P_{\mathcal{H}}(z_k) = \lambda_k x_k + \sum_{\ell=1}^{2^k} \eta_{k,\ell} \psi_{k,\ell},$$

for suitable scalars  $\lambda_k$  and  $\eta_{k,\ell}$ . Since from (3.3) we have  $x_k = 2^{-k/2} \sum_{\ell=1}^{2^k} \psi_{k,\ell}$ , we see that

$$c_{k,\ell} = 2^{-k/2} \lambda_k + \eta_{k,\ell}, \quad \ell = 1, \dots, 2^k. \quad (3.12)$$

We want to show that

$$\|P_{\mathcal{H}}(S_{\Lambda}z)\|^2 = \sum_{k=1}^{\infty} \|P_{\mathcal{H}}(S_{\Lambda}z_k)\|^2 \lesssim \|z\|^2 + \sum_{k=1}^{\infty} \|P_{\mathcal{H}}(z_k)\|^2, \quad (3.13)$$

from which (3.10) would follow easily, since the last series equals  $\|P_{\mathcal{H}}(z)\|^2 \leq \|z\|^2$ . To establish (3.13) we consider three possible situations for the index  $k$ ,

$$\begin{aligned} A_1 &= \{k : 2^{-k/2} |\lambda_k| < \alpha/2\} \\ A_2 &= \{k : 2^{-k/2} |\lambda_k| \in [\frac{\alpha}{2}, 2\alpha]\} \\ A_3 &= \{k : 2^{-k/2} |\lambda_k| > 2\alpha\}. \end{aligned}$$

Assume that  $k \in A_1$ . Then

$$\|P_{\mathcal{H}}(S_{\Lambda}z_k)\|^2 \leq \left\| \sum_{\ell \in \Lambda_k} c_{k,\ell} \psi_{k,\ell} \right\|^2 = \sum_{\ell \in \Lambda_k} |c_{k,\ell}|^2.$$

Now, when  $\ell \in \Lambda_k$  we have  $|c_{k,\ell}| \geq \alpha$ , and hence by (3.12)

$$\alpha \leq |2^{-k/2} \lambda_k + \eta_{k,\ell}| \leq \frac{\alpha}{2} + |\eta_{k,\ell}|.$$

Thus,  $\frac{\alpha}{2} \leq |\eta_{k,\ell}|$ , which in turn implies  $|c_{k,\ell}| \leq 2|\eta_{k,\ell}|$ . We conclude that, for  $k \in A_1$ ,

$$\|P_{\mathcal{H}}(S_{\Lambda}z_k)\|^2 \leq \sum_{\ell \in \Lambda_k} |c_{k,\ell}|^2 \leq 4 \sum_{\ell=1}^{2^k} |\eta_{k,\ell}|^2 = 4 \|P_{\mathcal{H}}(z_k)\|^2. \quad (3.14)$$

Next we consider  $k \in A_2$ . Here we use the cruder bound  $\|P_{\mathcal{H}}(S_{\Lambda}z_k)\| \leq \|S_{\Lambda}z_k\| \leq \|z_k\|$ , and notice that

$$\sum_{k \in A_2} \|z_k\|^2 = \sum_{k \in A_2} \left( \|P_{\mathcal{X}}(z_k)\|^2 + \|P_{\mathcal{H}}(z_k)\|^2 \right) = \sum_{k \in A_2} |\lambda_k|^2 + \sum_{k \in A_2} \|P_{\mathcal{H}}(z_k)\|^2.$$

We thus need to bound  $\sum_{k \in A_2} |\lambda_k|^2$ . Notice that  $A_2$  is a finite set (since  $2^{-k/2}\lambda_k \rightarrow 0$  as  $k \rightarrow \infty$ ), and write  $N_0 = \max A_2$ . Clearly,

$$2^{-N_0/2}|\lambda_{N_0}| \approx \alpha.$$

Then,

$$\sum_{k \in A_2} |\lambda_k|^2 \leq 4\alpha^2 \sum_{k \leq N_0} 2^k \leq 8\alpha^2 2^{N_0} \lesssim |\lambda_{N_0}|^2.$$

Since  $|\lambda_{N_0}| = \|P_{\mathcal{X}}(z_{N_0})\| \leq C\|z\|$ , we see that

$$\sum_{k \in A_2} \|P_{\mathcal{H}}(S_{\Lambda}z_k)\|^2 \lesssim \|z\|^2 + \sum_{k \in A_2} \|P_{\mathcal{H}}(z_k)\|^2. \quad (3.15)$$

Finally, consider  $k \in A_3$ . Using once again (3.12) we see that

$$S_{\Lambda}z_k = 2^{-\frac{k}{2}}\lambda_k \sum_{\ell \in \Lambda_k} \psi_{k,\ell} + \sum_{\ell \in \Lambda_k} \eta_{k,\ell} \psi_{k,\ell},$$

and therefore

$$\|P_{\mathcal{H}}(S_{\Lambda}z_k)\| \leq 2^{-\frac{k}{2}}|\lambda_k| \left\| P_{\mathcal{H}} \left( \sum_{\ell \in \Lambda_k} \psi_{k,\ell} \right) \right\| + \left( \sum_{\ell=1}^{2^k} |\eta_{k,\ell}|^2 \right)^{\frac{1}{2}}. \quad (3.16)$$

The second summand equals  $\|P_{\mathcal{H}}(z_k)\|$ , so we will work on the first.

Notice that  $P_{\mathcal{H}}(\sum_{\ell=1}^{2^k} \psi_{k,\ell}) = P_{\mathcal{H}}(2^{k/2}x_k) = 0$ , so we have

$$\left\| P_{\mathcal{H}} \left( \sum_{\ell \in \Lambda_k} \psi_{k,\ell} \right) \right\|^2 = \left\| P_{\mathcal{H}} \left( \sum_{\ell \in \Lambda_k^c} \psi_{k,\ell} \right) \right\|^2 \leq |\Lambda_k^c|. \quad (3.17)$$

Now,  $k \in A_3$  and  $\ell \in \Lambda_k^c$  imply  $|c_{k,\ell}| \leq \alpha$ . Using (3.12) we see that

$$2^{-\frac{k}{2}}|\lambda_k| - |\eta_{k,\ell}| \leq |2^{-\frac{k}{2}}\lambda_k + \eta_{k,\ell}| \leq \alpha \leq \frac{2^{-\frac{k}{2}}|\lambda_k|}{2},$$

and hence  $|\eta_{k,\ell}| \geq 2^{-k/2}|\lambda_k|/2$ . Therefore, using also (3.17), the middle term in (3.16) is bounded by

$$2^{-k}|\lambda_k|^2|\Lambda_k^c| \leq 4 \sum_{\ell \in \Lambda_k^c} |\eta_{k,\ell}|^2 \leq 4 \sum_{\ell=1}^{2^k} |\eta_{k,\ell}|^2 = 4\|P_{\mathcal{H}}(z_k)\|^2.$$

Thus, for  $k \in A_3$  in also have

$$\|P_{\mathcal{H}}(S_{\Lambda}z_k)\| \leq 3\|P_{\mathcal{H}}(z_k)\|. \quad (3.18)$$

Thus, we can now combine (3.14), (3.15) and (3.18) to obtain the asserted estimate in (3.13), and hence establish Lemma 3.4.  $\square$

**Lemma 3.5.** *The basis  $\Psi$  is democratic and, for every finite  $\Lambda$ ,*

$$\left\| \sum_{\lambda \in \Lambda} \psi_{\lambda} \right\| \approx |\Lambda|^{1/2}. \quad (3.19)$$

**PROOF:** The proof is a small refinement of the previous arguments. For simplicity, we use the notation  $\mathbf{1}_\Lambda = \sum_{\lambda \in \Lambda} \psi_\lambda$ , and set  $\Lambda_k = \{\lambda \in \Lambda : \lambda = (k, \ell) \text{ for some } \ell\}$ . Call  $N = |\Lambda|$  and  $N_k = |\Lambda_k|$ .

We first find an upper bound for

$$\|\mathbf{1}_\Lambda\|^2 = \|P_{\mathcal{X}}(\mathbf{1}_\Lambda)\|^2 + \|P_{\mathcal{H}}(\mathbf{1}_\Lambda)\|^2. \quad (3.20)$$

Arguing as in (3.11), the second term is easily estimated by

$$\|P_{\mathcal{H}}\mathbf{1}_\Lambda\|^2 = \sum_k \|P_{\mathcal{H}}(\mathbf{1}_{\Lambda_k})\|^2 \leq \sum_k \|\mathbf{1}_{\Lambda_k}\|^2 = \sum_k N_k = N.$$

For the first, since  $N_k \leq \min\{2^k, N\}$ , setting  $M = \log_2 N$ , and arguing as in (3.7)

$$\|P_{\mathcal{X}}\mathbf{1}_\Lambda\| = \left\| \sum_{k=1}^{\infty} 2^{-k/2} N_k x_k \right\| \leq \sum_{1 \leq k \leq M} 2^{k/2} + \sum_{k > M} 2^{-k/2} N \leq c \sqrt{N}.$$

We now find a lower bound for (3.20). Partition the indices  $k$  by

$$I_0 = \{k : N_k \leq 2^{k-1}\} \quad \text{and} \quad I_1 = \{k : 2^{k-1} < N_k \leq 2^k\}.$$

Let  $k_1 = \max I_1$ . Then, since  $N_k \approx 2^k$  for  $k \in I_1$ , we have

$$\|P_{\mathcal{X}}\mathbf{1}_\Lambda\|^2 = \left\| \sum_{k=1}^{\infty} 2^{-k/2} N_k x_k \right\|^2 \geq c 2^{-k_1} N_{k_1}^2 \geq c' \sum_{k \in I_1} N_k. \quad (3.21)$$

For the other term we use the identity

$$\|P_{\mathcal{H}}\mathbf{1}_{\Lambda_k}\|^2 = N_k (1 - 2^{-k} N_k). \quad (3.22)$$

Assuming (3.22), one sees that

$$\|P_{\mathcal{H}}\mathbf{1}_\Lambda\|^2 \geq \sum_{k \in I_0} \|P_{\mathcal{H}}(\mathbf{1}_{\Lambda_k})\|^2 \geq \frac{1}{2} \sum_{k \in I_0} N_k,$$

which combined with (3.21) gives  $\|\mathbf{1}_\Lambda\|^2 \gtrsim N$ . It remains to show (3.22), but this is easy, since by orthogonality

$$N_k = \|\mathbf{1}_{\Lambda_k}\|^2 = \|P_{\mathcal{X}}\mathbf{1}_{\Lambda_k}\|^2 + \|P_{\mathcal{H}}\mathbf{1}_{\Lambda_k}\|^2 = 2^{-k} N_k^2 + \|P_{\mathcal{H}}\mathbf{1}_{\Lambda_k}\|^2,$$

from which the claim follows easily.  $\square$

Finally we give a bound for  $k_N(\Psi)$  in terms of  $k_N(\mathfrak{X})$ .

**Lemma 3.6.** *Assume that the basis  $\mathfrak{X} = \{x_n\}$  of  $\mathcal{X}$  satisfies the property in (3.1). Then, the basis  $\Psi$  of  $\mathcal{X} \oplus \ell^2$  constructed above has*

$$k_M(\Psi) \geq c' k_{\log M}(\mathfrak{X}), \quad M = 2, 3, \dots \quad (3.23)$$

**PROOF:** Fix  $M$  and choose  $N$  such that  $2^{N-1} \leq M < 2^N$ . Select  $x$  and  $A$  as in (3.1), and set  $\Lambda = \cup_{k \in A} \{\psi_{k,1}, \dots, \psi_{k,2^k}\}$ , which has cardinality  $|\Lambda| \leq 2^{N+1}$ . Then

$$\|S_\Lambda x\| \geq \|P_{\mathcal{X}}S_\Lambda x\| = \|S_Ax\| \geq c k_N(\mathfrak{X}) \|x\| \geq c k_{\log_2 M}(\mathfrak{X}) \|x\|.$$

Since  $k_N$  is doubling, (3.23) follows easily.  $\square$

This completes the proof of Theorem 3.1.  $\square$

**3.2. Conditional bases with large  $k_N$ .** We shall use Theorem 3.1 to prove Theorem 1.2. For this purpose we first need to find a Hilbert space  $\mathcal{X}$  with a conditional basis  $\mathfrak{X}$  (not necessarily quasi-greedy) having  $k_N(\mathfrak{X})$  as large as possible. Here we give two examples in this direction.

We need the following elementary lemma about the Dirichlet kernel  $D_N(t) = \sum_{|n| \leq N} e^{int} = \sin(N + \frac{1}{2})t / \sin(\frac{t}{2})$ .

**Lemma 3.7.** *Let  $|\gamma| < 1$ . Then,*

$$\|D_N(t)\|_{L^2(|t|^\gamma dt)} \approx N^{\frac{1-\gamma}{2}}, \quad N = 1, 2, \dots \quad (3.24)$$

**PROOF:** The proof is elementary. From below,

$$\|D_N(t)\|_{L^2(|t|^\gamma dt)}^2 \geq \int_{|t| \leq 1/N} |D_N(t)|^2 |t|^\gamma dt \approx N^2 \int_{|t| \leq 1/N} |t|^\gamma dt \approx N^{1-\gamma}.$$

From above, the remaining part of the integral is estimated by

$$\int_{\frac{1}{N} \leq |t| < \pi} |D_N(t)|^2 |t|^\gamma dt \lesssim \int_{\frac{1}{N}}^\pi t^{\gamma-2} dt \lesssim N^{1-\gamma}. \quad \square$$

In the first example we obtain a *besselian* basis with  $k_N \gtrsim N^{\frac{1}{2}-\varepsilon}$ .

**Proposition 3.8.** *If  $\alpha \in (0, 1/2)$  then there is a Hilbert space  $\mathcal{X}$  with a besselian conditional basis  $\mathfrak{X} = \{x_n\}_{n=1}^\infty$  such that*

$$k_N(\mathfrak{X}) \gtrsim N^\alpha.$$

Moreover, for every  $N = 1, 2, \dots$  there exists a partition  $\{1, \dots, N\} = A \uplus B$  such that

$$\left\| \sum_{n \in A} x_n \right\|_{\mathcal{X}} \gtrsim N^\alpha \left\| \sum_{n \in A} x_n - \sum_{n \in B} x_n \right\|_{\mathcal{X}}. \quad (3.25)$$

**PROOF:** We consider the example proposed by Babenko [1]. That is, we set  $\mathcal{X} = L^2([-\pi, \pi], |t|^{-2\alpha} dt)$  with the usual trigonometric system  $\mathfrak{X} = \{1, e^{it}, e^{-it}, e^{2it}, e^{-2it}, \dots\}$ . That  $\mathfrak{X}$  is a basis follows from the fact that  $|t|^{-2\alpha}$  is an  $A_2$ -weight when  $|\alpha| < 1/2$  (see e.g. [7]). When  $\alpha > 0$  the weight is bounded from below by a positive constant, so  $\left\| \sum a_n e^{int} \right\|_{\mathcal{X}} \geq c \left\| \sum a_n e^{int} \right\|_{L^2} = c(\sum |a_n|^2)^{1/2}$ , and the basis is besselian.

We now prove (3.25). By the lemma

$$\left\| \sum_{|n| \leq N} e^{int} \right\|_{\mathcal{X}}^2 \gtrsim N^{1+2\alpha}.$$

On the other hand, Khintchine's inequality gives

$$\mathbb{E} \left[ \int_{\mathbb{T}} \left| \sum_{|n| \leq N} \pm e^{int} \right|^2 |t|^{-2\alpha} dt \right] \approx \int_{\mathbb{T}} N |t|^{-2\alpha} dt \approx N,$$

so for a certain fixed constant  $C > 0$ , there must exist some choice of signs  $\pm 1$  such that  $\left\| \sum_{|n| \leq N} \pm e^{int} \right\|_{\mathcal{X}} \leq C N^{1/2}$ . Partitioning  $\{-N, \dots, N\} = A_+ \uplus A_-$  according to these signs, we have shown that

$$\left\| \sum_{|n| \leq N} e^{int} \right\|_{\mathcal{X}} \gtrsim N^\alpha \left\| \sum_{n \in A_+} e^{int} - \sum_{n \in A_-} e^{int} \right\|_{\mathcal{X}}.$$

Using that  $\left\| \sum_{|n| \leq N} e^{int} \right\|_{\mathcal{X}} \leq 2 \left\| \sum_{n \in A} e^{int} \right\|_{\mathcal{X}}$  either for  $A = A_+$  or  $A = A_-$ , (3.25) follows easily.  $\square$

**Remark 3.9.** The example in Proposition 3.8 actually satisfies  $k_N \approx N^\alpha$ . We sketch a proof of the upper bound in Appendix I.

In our second example we find a basis in a Hilbert space with  $k_N \gtrsim N^{1-\varepsilon}$ . It is a consequence of a well-known theorem of Gurarii and Gurarii [10] that this growth is best possible.

**Proposition 3.10.** *Let  $\alpha < 1$ . Then there is a Hilbert space  $\mathcal{X}$  with a conditional basis  $\mathfrak{X} = \{x_n\}_{n=1}^\infty$  such that  $k_N(\mathfrak{X}) \gtrsim N^\alpha$ . Moreover, there exists  $c > 0$  such that for every  $N = 1, 2, \dots$  there is a set  $A \subset \{1, \dots, N\}$  and a non-null  $x \in \mathcal{X}$  so that*

$$\|S_A(x)\|_{\mathcal{X}} \geq c N^\alpha \|x\|_{\mathcal{X}}. \quad (3.26)$$

**PROOF:** Consider  $\mathcal{X} = L^2(\mathbb{T}, |t|^\alpha dt) \oplus L^2(\mathbb{T}, |t|^{-\alpha} dt)$  for the  $\ell^2$ -direct sum. Call  $\{e_k\}_{k=1}^\infty$  and  $\{f_k\}_{k=1}^\infty$  the respective trigonometric bases as in the previous proposition, and consider a new basis in  $\mathcal{X}$  given by

$$x_{2k-1} = \frac{e_k + f_k}{\sqrt{2}}, \quad x_{2k} = \frac{e_k - f_k}{\sqrt{2}}, \quad k = 1, 2, \dots \quad (3.27)$$

For  $N \geq 1$ , let  $x = \sum_{n=1}^{2N} x_n = \sqrt{2} \sum_{k=1}^N e_k$ . If  $N = 2m + 1$ , then  $x$  coincides with the Dirichlet kernel  $D_m(t)$ , which by Lemma 3.7 gives the estimate

$$\|x\|_{\mathcal{X}} = \sqrt{2} \|D_m(t)\|_{L^2(|t|^\alpha dt)} \approx N^{\frac{1-\alpha}{2}}.$$

Now let  $A = \{1, 3, \dots, 2N - 1\}$ , so that  $S_A(x) = \frac{1}{\sqrt{2}} \sum_{k=1}^N e_k + \frac{1}{\sqrt{2}} \sum_{k=1}^N f_k$ , which has norm

$$\|S_A(x)\|_{\mathcal{X}} \geq \frac{1}{\sqrt{2}} \left\| \sum_{k=1}^N f_k \right\| = \frac{1}{\sqrt{2}} \|D_m(t)\|_{L^2(|t|^{-\alpha} dt)} \approx N^{\frac{1+\alpha}{2}}.$$

Combining these two estimates we obtain  $k_N \geq \|S_A(x)\|/\|x\| \gtrsim N^\alpha$ , as well as the assertion in (3.26).  $\square$

**3.3. End of the proof of Theorem 1.2.** Fix  $\alpha < 1$  and apply Theorem 3.1 to the Hilbert space  $\mathcal{X}$  and the basis  $\mathfrak{X} = \{x_n\}$  in Proposition 3.10. This produces a quasi-greedy basis  $\Psi$  in the Hilbert space  $\mathcal{X} \oplus \ell^2$ . Since the assumptions of Lemma 3.6 hold with  $k_N(\mathfrak{X}) \approx N^\alpha$  (by (3.26)), we obtain that

$$k_M(\Psi) \geq c_\alpha (\log M)^\alpha, \quad M = 1, 2, \dots$$

as we wished to prove.

If we only assume  $\alpha < 1/2$ , then we would argue similarly, using instead Proposition 3.8, so that the basis  $\Psi$  of  $\mathcal{X} \oplus \ell^2$  is in addition besselian. In this case, the dual system to  $\Psi$  will be a hilbertian quasi-greedy basis for  $(\mathcal{X} \oplus \ell^2)^*$  with the same bound on  $k_M$ .  $\square$

**3.4. Further results.** As a consequence of Theorem 3.1 we can find new examples of quasi-greedy democratic bases in Banach spaces for which  $k_N \approx \log N$  (see also [8]).

**Corollary 3.11.** *There exists a basis in  $c_0 \oplus \ell^1 \oplus \ell^2$  which is quasi-greedy, democratic and has  $k_N \approx \log N$ .*

**PROOF:** Consider  $c_0$  and  $\ell^1$  with their usual canonical bases, say  $\{e_k\}$  and  $\{f_k\}$ , and define in  $\mathcal{X} = c_0 \oplus \ell^1$  a basis  $\mathfrak{X} = \{x_n\}$  as in (3.27). Then a similar proof as in Proposition 3.10 shows that  $k_N(\mathfrak{X}) \gtrsim N$  and (3.26) holds with  $\alpha = 1$ . Then, Theorem 3.1 gives a quasi-greedy democratic basis  $\Psi$  in  $\mathcal{X} \oplus \ell^2 = c_0 \oplus \ell^1 \oplus \ell^2$  with  $k_N(\Psi) \gtrsim \log N$ .  $\square$

Theorem 3.1 can also be used to show the optimality of the bound  $k_N = O(\log N)^\alpha$  with  $\alpha < 1$  for quasi-greedy bases in  $\ell^p$  or  $L^p$  spaces,  $1 < p < \infty$  (see Theorem 5.1 below).

**Corollary 3.12.** *Let  $1 < p < \infty$  and  $\alpha < 1$ . Then, there exists a quasi-greedy basis in  $\ell^p$  with  $k_N \gtrsim (\log N)^\alpha$ .*

**PROOF:** Use a similar construction to [18, Corollary 5].  $\square$

In the spaces  $L^p[0, 1]$  more can be said. Namely, a direct application of Theorem 3.1 gives bases which are additionally democratic, with democracy function  $N^{1/2}$ .

**Corollary 3.13.** *Let  $1 < p < \infty$  and  $\alpha < 1$ . Then, there exists a quasi-greedy democratic basis  $\Psi$  in  $L^p[0, 1]$  with  $k_N(\Psi) \gtrsim (\log N)^\alpha$  and  $\|\sum_{\lambda \in \Lambda} \psi_\lambda\| \approx |\Lambda|^{1/2}$ .*

**PROOF:** We use the isomorphism  $L^p \simeq L^p \oplus \ell^2$ . This isomorphism already implies that there is a conditional basis  $\mathfrak{X}$  in  $L^p$  with  $k_N(\mathfrak{X}) \gtrsim N^\alpha$ . Indeed, just take any unconditional basis  $\mathcal{B}_0$  in  $L^p$  and a conditional basis  $\mathcal{B}_1$  in  $\ell^2$  with  $k_N(\mathcal{B}_1) \gtrsim N^\alpha$  (using e.g. Proposition 3.10), and consider in  $L^p \oplus \ell^2$  the joint basis  $\mathfrak{X} = \mathcal{B}_0 \cup \mathcal{B}_1$  (say in alternate order). This basis will satisfy  $k_N(\mathfrak{X}) \gtrsim N^\alpha$  and also (3.1).

Now use that  $L^p \simeq (L^p \oplus \ell^2) \oplus \ell^2$ , and apply the construction in Theorem 3.1, using the basis  $\mathfrak{X}$  in the first summand  $\mathcal{X} = L^p \oplus \ell^2$ , to obtain a new basis  $\Psi$  with the required properties.  $\square$

#### 4. APPENDIX I

We show the claim asserted in Remark 3.9.

**Proposition 4.1.** *For  $\alpha \in (0, \frac{1}{2})$ , consider the Hilbert space  $L^2([-\pi, \pi], |t|^{-2\alpha} dt)$  with the trigonometric basis  $\{1, e^{it}, e^{-it}, e^{2it}, e^{-2it}, \dots\}$ . Then*

$$k_N \lesssim N^\alpha, \quad N = 1, 2, \dots$$

We shall use the following lemma.

**Lemma 4.2.** *Let  $\gamma \in (0, 1)$  and  $\omega(t) = |t|^{-\gamma}$ ,  $|t| < \pi$ . Then its Fourier coefficients satisfy*

$$|\widehat{\omega}(k)| \leq \frac{c_\gamma}{|k|^{1-\gamma}}, \quad k \in \mathbb{Z} \setminus \{0\}. \quad (4.1)$$

**PROOF:** We use the method of stationary phase. Let  $\delta = 1/|k|$ . Then

$$\left| \int_0^\delta e^{-ikt} t^{-\gamma} dt \right| \leq \int_0^\delta t^{-\gamma} dt = \frac{\delta^{1-\gamma}}{1-\gamma} = \frac{c_\gamma}{|k|^{1-\gamma}}.$$

On the other hand, by Van der Corput's Lemma [14, p. 334]

$$\left| \int_\delta^\pi e^{-ikt} |t|^{-\gamma} dt \right| \leq \frac{c}{|k|} \left( \frac{1}{\pi^\gamma} + \int_\delta^\pi \gamma |t|^{-\gamma-1} dt \right) = \frac{c}{|k| \delta^\gamma} = \frac{c}{|k|^{1-\gamma}}.$$

These two estimates easily imply (4.1).

□

**PROOF of Proposition 4.1:** Let  $|\Lambda| = N$ . It suffices to show that

$$\left\| \sum_{\lambda \in \Lambda} a_\lambda e^{i\lambda t} \right\|_{L^2(|t|^{-2\alpha} dt)}^2 \lesssim N^{2\alpha} \sum_{\lambda} |a_\lambda|^2, \quad (4.2)$$

since the basis is besselian. We can write the left hand side as

$$LHS = \int_{-\pi}^{\pi} \sum_{\lambda, \mu \in \Lambda} a_\lambda \bar{a}_\mu e^{i(\lambda-\mu)t} |t|^{-2\alpha} dt = \sum_{\lambda, \mu \in \Lambda} a_\lambda \bar{a}_\mu \widehat{\omega}(\mu - \lambda).$$

Using Lemma 4.2 (with  $\gamma = 2\alpha$ ) we obtain

$$\begin{aligned} LHS &\lesssim \sum_{\lambda} |a_\lambda|^2 + \sum_{\lambda \neq \mu \in \Lambda} \frac{|a_\lambda| |a_\mu|}{|\mu - \lambda|^{1-2\alpha}} \\ &\leq \sum_{\lambda} |a_\lambda|^2 + \left( \sum_{\lambda} |a_\lambda|^2 \right)^{\frac{1}{2}} \left[ \sum_{\lambda \in \Lambda} \left( \sum_{\substack{\mu \in \Lambda \\ \mu \neq \lambda}} \frac{|a_\mu|}{|\mu - \lambda|^{1-2\alpha}} \right)^2 \right]^{\frac{1}{2}}. \end{aligned} \quad (4.3)$$

Now,

$$\sup_{\lambda \in \Lambda} \sum_{\substack{\mu \in \Lambda \\ \mu \neq \lambda}} \frac{1}{|\mu - \lambda|^{1-2\alpha}} \leq \sum_{j=1}^{|\Lambda|} \frac{1}{j^{1-2\alpha}} \leq cN^{2\alpha},$$

so Schur's lemma (or Cauchy-Schwarz) easily leads from (4.3) to (4.2). □

## 5. APPENDIX II

The following is a variation of Theorem 1.1 for  $L^p$  spaces. Throughout this section we fix  $1 < p < \infty$ , and let  $\|\cdot\|$  stand for the usual norm in  $L^p(X, \mu)$ . We denote by  $\kappa = \kappa(p)$  the smallest constant such that

$$\|G_N x\| \leq \kappa \|x\| \quad \text{and} \quad \|x - G_N x\| \leq \kappa \|x\|, \quad \forall x \in L^p, \quad N = 1, 2, \dots \quad (5.1)$$

**Theorem 5.1.** *Let  $1 < p < \infty$  and  $\{\mathbf{e}_j\}$  a quasi-greedy (normalized) basis in  $L^p$ . Then, there exists  $\alpha = \alpha(\kappa, p) < 1$  and  $c > 0$  such that*

$$k_N(\{\mathbf{e}_j\}) \leq c (\log N)^\alpha, \quad \forall N \in \mathbb{N}. \quad (5.2)$$

The result depends on an  $L^p$ -version of Lemma 2.2, which we state with constants that very likely are not optimal.

**Lemma 5.2.**

(i) *If  $1 < p \leq 2$  then for  $c_p = 2 - \frac{p-1}{2\kappa^2}$  it holds*

$$\|x + y\|^2 \leq c_p \left( \|x\|^2 + \|y\|^2 \right), \quad \forall x \succsim y. \quad (5.3)$$

(ii) *If  $2 \leq p < \infty$  then for  $c_p = 2^{p-1} - \frac{1}{2\kappa^p}$  it holds*

$$\|x + y\|^p \leq c_p \left( \|x\|^p + \|y\|^p \right), \quad \forall x \succsim y. \quad (5.4)$$

To prove Lemma 5.2 we shall use weak versions of the parallelogram identity which are well-known in the literature

$$\|x + y\|^2 + (p - 1)\|x - y\|^2 \leq 2(\|x\|^2 + \|y\|^2), \quad 1 < p \leq 2 \quad (5.5)$$

$$\|x + y\|^p + \|x - y\|^p \leq 2^{p-1}(\|x\|^p + \|y\|^p), \quad 2 \leq p < \infty. \quad (5.6)$$

The first one appears in work of Bynum and Drew [2], and the second one is attributed to Clarkson [3].

**Case 1**  $1 < p \leq 2$ . Call  $N = |\text{supp } x|$ . The assumption  $x \succcurlyeq y$  and (5.1) give

$$\|x\| = \|G_N(x - y)\| \leq \kappa \|x - y\| \quad \text{and} \quad \|y\| = \|(I - G_N)(x - y)\| \leq \kappa \|x - y\|. \quad (5.7)$$

Thus,

$$\|x - y\|^2 \geq \frac{1}{2\kappa^2} (\|x\|^2 + \|y\|^2).$$

Now the weak parallelogram law in (5.5) combined with the previous estimate gives

$$\|x + y\|^2 \leq 2(\|x\|^2 + \|y\|^2) - (p - 1)\|x - y\|^2 \leq \left(2 - \frac{p-1}{2\kappa^2}\right) (\|x\|^2 + \|y\|^2),$$

which proves (5.3).

We now sketch the proof of Theorem 5.1 in this case. Arguing in exactly the same way as in the proof of Theorem 1.1 one reaches the inequality (2.7), this time with  $(1 + \delta)$  replaced by the constant  $c_p$ . We follow the second approach alluded in that proof, obtaining (2.8), and hence the validity of (2.5) with  $\alpha = (1 + \log_2 c_p)/2 < 1$ . This establishes (5.2) when  $1 < p \leq 2$ .

**Case 2**  $2 \leq p < \infty$ . We first establish (5.4). From (5.7) observe that

$$\|x - y\|^p \geq \frac{1}{2\kappa^p} (\|x\|^p + \|y\|^p).$$

This, combined with Clarkson's inequality in (5.6) gives

$$\|x + y\|^p \leq 2^{p-1}(\|x\|^p + \|y\|^p) - \|x - y\|^p \leq \left(2^{p-1} - \frac{1}{2\kappa^p}\right) (\|x\|^p + \|y\|^p).$$

Now one proceeds as in the proof of Theorem 1.1, but with powers 2 replaced by powers  $p$ , and  $(1 + \delta)$  replaced by  $c_p$ . Then the corresponding version of (2.8) leads to

$$\|S_A(x)\| \lesssim c_p^{(\log_2 m)/p} m^{1/p} \|x\| = m^{(1+\log_2 c_p)/p} \|x\|.$$

So we can set  $\alpha = (1 + \log_2 c_p)/p$ , which is smaller than 1 since  $c_p < 2^{p-1}$ .  $\square$

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